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Bifurcation of vortex lines in Euler flow

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Abstract
A class of Euler flows of an ideal incompressible liquid is considered. The total kinetic helicity is invariant for barotropic inviscid flow under conservative body forces. The topological structure of vortex lines are classified by Hopf indices, Brouwer degrees and linking number in geometry. A new mechanism of generation and annihilation of a vortex line is given. The evolution equation of the vortex line has been given and its splitting behavior at the critical points is also discussed in detail. Three length approximation relations in the neighbourhood of singular points are given: $l \propto (t - t^*)^{1/2}, l \propto t - t^*$, $l = \text{const}$.

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1. Introduction

Vortex dynamics plays an important role in airfoils [1], fluid dynamics [2], magnetohydrodynamic [3, 4], samll scale turbulence and astrophysics [5, 6]. The vorticity field is a solenoidal field and will not have the field line with end points within the flow. Thus, it is convenient to study the evolution of vortex lines in terms of certain topological indicators of closed curves. The most important topological invariant for the vortex lines is the kinetic helicity, which is a topological invariant for barotropic inviscid flow under conservative body forces [7]. The kinetic helicity results from Kelvin theorem on circulation and measures the entangledness of the vortex lines. It is the simplest measure of topological complexity of an advected fluid. It characterizes the internal structure of the vortex tubes (twisting, torsion and kinking) and also the external relationships among the tubes themselves, such as linking and knotting of vortex tubes. Helical flow structures exist in nature where free shear flows occur, such as in tornadoes and storm systems. Helical modes are also known to be important in the wakes of axisymmetric bodies when the angle of attack is nonzero. Helical structures can spontaneously emerge from nonhelical (mirror symmetric) states due to the growth of unstable modes. Such breakdown of the mirror symmetry can occur in a rotating flow since the rotation vector provides a preferred direction and can lead from a nonhelical state to a helical flow. This can be of primary importance for the α dynamo effect [8] where helical fluctuations can, under certain conditions, amplify the mean magnetic field.

It is known that the equations of an ideal incompressible liquid, i.e. the Euler equations,

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V} + \nabla p = 0, \qquad \operatorname{div} \mathbf{V} = 0, \tag{1}$$

are Hamiltonian equations [9]. The Hamiltonian structure can be easily introduced in terms of vorticity $\mathbf{\Omega} = \operatorname{rot} \mathbf{V}$, determined from the equation

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \operatorname{rot}[\mathbf{V}, \mathbf{\Omega}], \tag{2}$$

where the square brackets denote the vector product of the two-vector velocity and vorticity. In this case,

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \{\mathbf{\Omega}, H\}, \tag{3}$$

where the Hamiltonian H is the energy of the system,

$$H = \int \frac{1}{2} \mathbf{V}^2 \, \mathrm{d}^3 x, \tag{4}$$

and the Poisson brackets for any two functions F and G are defined by

$$\{F, G\} = \int \left(\mathbf{\Omega}, \left[\operatorname{rot} \frac{\delta F}{\delta \mathbf{\Omega}}, \operatorname{rot} \frac{\delta G}{\delta \mathbf{\Omega}} \right] \right) \mathrm{d}^3 x. \tag{5}$$

Here, $\delta F/\delta\Omega$, $\delta G/\delta\Omega$ are variational derivatives, and round brackets are defined as a dot product. The given form possesses all the necessary properties of Poisson brackets. This form is antisymmetric and satisfies the Jacobi identity. Hence, equation (2) is a Hamiltonian equation.

The liquid flow topology can be characterized by a kinetic helicity Γ as

$$\Gamma = \int (\mathbf{V}, \mathbf{\Omega}) \, \mathrm{d}^3 x. \tag{6}$$

The kinetic helicity Γ is an invariant [7] for both incompressible and compressible polytropic nonmagnetized flows in conservative forces and in a compact domain, which is a direct consequence of the Thompson theorem [10].

In the present work, we consider a class of Euler flows of an ideal incompressible liquid and focus on the kinetic helicity. In section 2, we classify the topological structure of the vortex lines in terms of Hopf index, Brouwer degree and linking number in geometry. We discuss the evolution equation of the vortex line in terms of **n**-field [11]. In section 3, we give a new mechanism of generation and annihilation of vortex lines. In section 4, we study the bifurcation [12] behavior of Euler flow at the bifurcation point in detail. There are four cases and three kinds of length approxiamtion relation.

2. Topological structure of the vortex lines

Following Faddeev [11], the transverse field $\mathbf{\Omega}$ can be expressed in terms of the **n**-field [13]

$$\Omega^i = A\epsilon^{ijk}(\mathbf{n} \cdot [\partial_j \mathbf{n}, \partial_k \mathbf{n}]), \qquad i, j, k = 1, 2, 3, \tag{7}$$

where $\mathbf{n}^2 = 1$, A is a dimensional constant that does not depend on the time or the coordinates. Volovik and Mineev have shown [14] that for the quantum case, $A = \hbar/4m$. The above formula gives the transition from a differential relationship between the components of the vorticity field div $\mathbf{\Omega} = 0$ to an algebraic one $\mathbf{n}^2 = 1$. For the given class of flows, R^3 is isoporphic to S^3, i.e., the problem of classification of flow is that of classification of smooth maps $S^3 \rightarrow S^2$. These maps are characterized by a homotropic group $\pi_3(S^2) = Z$, i.e. any class of flows is determined by the integer values that represent the linking number for the two curves $\mathbf{n}(r) = \mathbf{n}_1$ and $\mathbf{n}(r) = \mathbf{n}_2$, and consequently, for the two vortex lines corresponding to these curves. This index for smooth maps $S^3 \rightarrow S^2$ is called the Hopf invariant which can be expressed via the map $\mathbf{n}(r)$ [15]. The unit vector field $\mathbf{n}$ is a section of sphere bundle S^2.

We define two two-dimensional unit vector $\mathbf{e}_1, \mathbf{e}_2$ in S^2, which are normal to each other, i.e.,

$$\begin{aligned} &\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{n} = 0 \\ &\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{n} \cdot \mathbf{n} = 1. \end{aligned} \tag{8}$$

It is easily obtained that $\mathbf{n} \cdot [\partial_j \mathbf{n}, \partial_k \mathbf{n}] = 2\epsilon^{ab}\partial_j e_1^a \partial_k e_2^b$. Then the velocity field $\mathbf{V}$ can be written as [11, 16]

$$\mathbf{V} = 2A\mathbf{e}_1 \cdot \nabla \mathbf{e}_2. \tag{9}$$

Let us consider a two-dimensional order parameter $\psi = (\psi^1, \psi^2)$ in a plane formed by unit vectors e_1, e_2, which satisfies

$$e_1^a = \frac{\psi^a}{\|\psi\|}, \qquad e_2^a = \epsilon^{ab}\frac{\psi^a}{\|\psi\|}, \qquad a, b = 1, 2, \tag{10}$$

where $\|\psi\| = (\psi^a\psi^a)^{1/2}$, and ϵ is the Levi-Civita antisymmetric tensor. The zero points of the order parameter are just the singular points of e_1 and e_2. The velocity $\mathbf{V}$ can be expressed by

$$\mathbf{V} = 2A\epsilon^{ab}\frac{\psi^a}{\|\psi\|}\nabla\frac{\psi^b}{\|\psi\|}. \tag{11}$$

The transverse field can be written now in terms of the ψ field

$$\Omega^i = 2A\epsilon^{ijk}\epsilon_{ab}\partial_j\frac{\psi^a}{\|\psi\|}\partial_k\frac{\psi^b}{\|\psi\|}. \tag{12}$$

Using the relation

$$\partial_b\frac{\psi^a}{\|\psi\|} = \frac{\partial_b\psi^a}{\|\psi\|} - \frac{\psi^a\psi^b}{\|\psi\|^3}, \qquad \partial_a\partial_a \ln\|\psi\| = 2\pi\delta^2(\psi), \tag{13}$$

the transverse field becomes

$$\Omega^i = 8\pi A\delta^2(\psi)\mathrm{D}^i\left(\frac{\psi}{x}\right), \tag{14}$$

where [17, 18]

$$\mathrm{D}^i\left(\frac{\psi}{x}\right) = \frac{1}{2}\epsilon^{ijk}\epsilon^{ab}\partial_j\psi^a\partial_k\psi^b, \qquad i, j, k = 1, 2, 3, \quad a, b = 1, 2. \tag{15}$$

Equation (15) tells us that the transverse field,

$$\begin{aligned} &\Omega^i = 0 \qquad \text{only if } \psi \neq 0, \\ &\Omega^i \neq 0 \qquad \text{only if } \psi = 0. \end{aligned} \tag{16}$$

In appendix A, we calculate it in detail. Then we can obtain

$$\int_{M_k} \Omega^i \, \mathrm{d}\sigma_i = 8\pi A\beta_k\eta_k. \tag{17}$$

Here σ is the surface surrounded by the vortex. Substituting equation (17) into equation (6), one can obtain

$$\Gamma = 8\pi A \sum_{k=1}^{N} \beta_k \eta_k \int_{L_k} V_i \, \mathrm{d}x^i. \tag{18}$$

When these vortex lines are closed curves, i.e. a family of knots $\xi_k (k = 1, 2, \ldots, N)$, equation (18) becomes

$$\Gamma = 8\pi A \sum_{k=1}^{N} \beta_k \eta_k \oint_{\xi_k} V_i \, \mathrm{d}x^i. \tag{19}$$

In appendix B, we have calculated equation (19) in detail. Then we obtain the important result

$$\Gamma = 64\pi^2 A^2 \left[\sum_{k=1}^{N} \beta_k \eta_k S(\xi_k) + \sum_{k,l=1}^{N} \beta_k \eta_k L(\xi_k, \xi_l) \right]. \tag{20}$$

The first term is the self-linking number $S(\xi_k)$ of the vortex line ξ_k; the second term is the Gauss linking number L of the vortex lines ξ_k and ξ_l. We denote the total topological number C of vortex lines configuration as

$$C = \sum_{k=1}^{N} \beta_k \eta_k S(\xi_k) + \sum_{k,l=1}^{N} \beta_k \eta_k L(\xi_k, \xi_l), \tag{21}$$

which is a Hopf invariant, and is also called a topological charge by Faddeev. Then

$$\Gamma = 64\pi^2 A^2 C. \tag{22}$$

This result is correct in either quantum case [14] or classical fluid [7]. It is obvious that $8\pi A \beta_k \eta_k$ $(A = \hbar/4m)$ in the quantum case corresponding to the classical flux strength χ of the vortex. If there are N filaments with strength χ_k $(k = 1, 2, \ldots, N))$ whose self-knottedness degree, i.e. $\beta_k = 1$ in a classical fluid, the kinetic helicity equals

$$64\pi^2 A^2 \sum_{k,l=1}^{N} \eta_k L(\xi_k, \xi_l) = \sum_{k,l=1}^{N} \chi_k \chi_l \eta_k \eta_l \alpha_{kl}$$

($\alpha_{kl} = 1$ if two vortex lines ξ_k, ξ_l are linked, $\alpha_{kl} = 0$ if ξ_k, ξ_l are not singly linked). The kinetic helicity is an invariant for both incompressible and compressible polytropic nonmagnetized flows in conservative forces and in a compact domain. In the next two sections we will discuss bifurcation behavior of vortex lines in Euler flow, which keep the kinetic helicity invariant.

In this section, the topological structure of the vortex line is studied under the regular condition, i.e., $\mathrm{D}(\psi/x) \neq 0$. When the regular condition fails, the branching of vortex line will occur. This will be discussed in sections 3 and 4.

3. Branching of vortex lines

The evolution equation of the vector field $\mathbf{n}$ has been obtained [13] by Kuznetsov *et al* i.e.,

$$\frac{\partial \mathbf{n}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{n} = 0. \tag{23}$$

It is also a Hamiltonian:

$$\frac{\partial \mathbf{n}}{\partial t} = \left[\mathbf{n}, \frac{\delta (H/A)}{\delta \mathbf{n}} \right]. \tag{24}$$

The evolution of the vortex line can be discussed from equation (14). For simplicity, we fix the $x^3 = z$ coordinate and take the XOY plane as the cross section. The intersection line between the vortex line evolution surface and the cross section is just the motion curve of the vortex line. In this two-dimensional case, the vorticity becomes [18]

$$\Omega^3 = 8\pi\delta^2(\psi)\mathrm{D}\left(\frac{\psi}{x}\right) \tag{25}$$

and

$$\Omega^i = 8\pi\delta^2(\psi)\mathrm{D}^i\left(\frac{\psi}{x}\right), \qquad i = 1, 2. \tag{26}$$

where $\mathrm{D}\left(\frac{\psi}{x}\right) = \epsilon^{ab}\partial_1\psi^a\partial_2\psi^b$, $\mathrm{D}^1\left(\frac{\psi}{x}\right) = \epsilon^{ab}\partial_2\psi^a\partial_t\psi^b$, $\mathrm{D}^2\left(\frac{\psi}{x}\right) = \epsilon^{ab}\partial_t\psi^a\partial_1\psi^b$.

It is obvious that the continuity equation is satisfied:

$$\partial_t\Omega^3 + \partial_i\Omega^i = 0. \tag{27}$$

The velocity of the intersection point of the vortex line and the cross section is given as

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{D}^i\left(\frac{\psi}{x}\right)}{\mathrm{D}\left(\frac{\psi}{x}\right)}. \tag{28}$$

From equation (28), we know that when $\mathrm{D}\left(\frac{\psi}{x}\right) = 0$ at the very point $(t^*, \mathbf{x}^*)$, the velocity $\frac{\mathrm{d}x^1}{\mathrm{d}t}$ or $\frac{\mathrm{d}x^2}{\mathrm{d}t}$ is not unique in the neighborhood of (t^*, x^*). This critical point is called the branch point [19, 21], which is also called the singularity point by Kerr *et al*. At the critical point, the normal velocity can not be defined, which is also pointed out by other physicists [3, 18]. Because of the conservation of vortex circulation, it should branch or split [19, 20]. Taking the Taylor expansion of the solution of ψ at the critical point, one can obtain the direction of the zero point on the cross section at the critical point. Let us do that in the following. If we assume that $\mathrm{D}^2\left(\frac{\psi}{x}\right)_{(t^*,\mathbf{x}^*)} \neq 0$, then there are usually two kinds of branch points, namely the limit points where $\mathrm{D}^1\left(\frac{\psi}{x}\right)\big|_{(t^*,\mathbf{x}^*)} \neq 0$ and the bifurcation points where $\mathrm{D}^1\left(\frac{\psi}{x}\right)_{(t^*,\mathbf{x}^*)} = 0$. In this section, we discuss only the branching process of the vortex lines at the limit point. When $\mathrm{D}^1\left(\frac{\psi}{x}\right)\big|_{(t^*,\mathbf{x}^*)} \neq 0$, we obtain from equation (28)

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = \left.\frac{\mathrm{D}^1\left(\frac{\psi}{x}\right)}{\mathrm{D}\left(\frac{\psi}{x}\right)}\right|_{(t^*,\mathbf{x}^*)} = \infty, \tag{29}$$

i.e.,

$$\left.\frac{\mathrm{d}t}{\mathrm{d}x^1}\right|_{(t^*,\mathbf{x}^*)} = 0. \tag{30}$$

Taking the Taylor expansion of $t = t(x^1, t)$ at the limit point of the vortex line, one can obtain

$$t - t^* = \frac{1}{2}\left.\frac{\mathrm{d}^2t}{(\mathrm{d}x^1)^2}\right|_{(t^*,\mathbf{x}^*)}(x^1 - x^{1*})^2, \tag{31}$$

which is a parabola in $x^1 - t$ plane. From equation (31) one can obtain two solutions, which give the branch solutions of the vortex line at the limit points. If $\frac{\mathrm{d}^2t}{(\mathrm{d}x^1)^2}\big|_{(t^*,\mathbf{x}^*)} > 0$, we have the branch solutions for $t > t^*$; otherwise, we have the branch solutions for $t < t^*$. The former is related to the origin of the vortex line at the limit points. From the continuity equation, we know that the topological number of the vortex line is identically conserved. This means that the total topological number of the final vortex lines equals that of the initial vortex lines. The total numbers of these two generated vortex lines must be zero at the limit point, i.e. the two generated vortex lines have to be opposite, i.e.,

$$\beta_1\eta_1 + \beta_2\eta_2 = 0. \tag{32}$$

It is a process of generation or annihilation of vortex lines [22–24]. At the neighborhood of the limited point, we denote length scale $l = \Delta x = x - x^*$, $\Delta t = t - t^*$. From equation (31), one can obtain the approximation relation

$$l \propto \|t - t^*\|^{1/2}. \tag{33}$$

The growth rate $\gamma = \frac{l}{\Delta t}$ or the annihilation rate of the vortex lines

$$\gamma \propto (t - t^*)^{-1/2}. \tag{34}$$

It is obvious that $E_k \propto (t - t^*)^{-1}$ [25]. This result is obtained in the neighborhood of the limited point. Then it is correct locally. This result agrees with the numerical data [26, 27].

4. Bifurcation of vortex lines

Now let us study the bifurcation of vortex line at its bifurcation point where $D^1\left(\frac{\psi}{x}\right)\big|_{(t^*,\mathbf{x}^*)} = 0$. The Taylor expansion of the solution of ψ^1 and ψ^2 in the neighborhood of the bifurcation point can generally be denoted as $F(x^1 - x^{1*})^2 + 2B(x^2 - x^{2*})(t - t^*) + C(t - t^*)^2 + \cdots = 0$, where F, B and C are three constants. Here we assume that $F \neq 0$; then from the Taylor expansion, we can obtain

$$F\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + C = 0. \tag{35}$$

There are four kinds of important cases.

Case 1. $F \neq 0$, $(B^2 - FC) > 0$. We get two different directions of vortex lines

$$\left.\frac{dx^1}{dt}\right|_{(t^*,\mathbf{x}^*)} = \frac{-B \pm \sqrt{B^2 - FC}}{F}. \tag{36}$$

It is the intersection of two vortex lines, which means that the two vortex lines meet and then depart from each other at the bifurcation point.

Case 2. $F \neq 0$, $(B^2 - FC) = 0$. The direction of the vortex lines is the only one,

$$\left.\frac{dx^1}{dt}\right|_{(t^*,\mathbf{x}^*)} = \frac{-B}{F}, \tag{37}$$

which includes three important situations: (a) one vortex line splits into three vortex lines; (b) two vortex lines merge into one vortex line and (c) two vortex lines tangentially intersect at the bifurcation point.

Case 3. $F = 0$, $(B^2 - FC) \neq 0$ (or $B \neq 0$), $C \neq 0$. We have

$$\left.\frac{dt}{dx^1}\right|_{(t^*,\mathbf{x}^*)} = \frac{-B \pm \sqrt{B^2 - FC}}{C} = 0, \qquad -\frac{2B}{C}. \tag{38}$$

There are two important cases: first, one vortex line splits into three at the bifurcation point; second, three vortex lines merge into one at the bifurcation point.

Case 4. $F = C = 0$. We obtain

$$\left.\frac{dt}{dx^1}\right|_{(t^*,\mathbf{x}^*)} = 0, \qquad \text{or} \qquad \left.\frac{dx^1}{dt}\right|_{(t^*,\mathbf{x}^*)} = 0. \tag{39}$$

At the neighborhood of the bifurcation point, we denote the scale length as $\Delta x = l$. From equations (36)–(38), we can then obtain the approximation asymptotic relation

$$l \propto (t - t^*). \tag{40}$$

The growth rate γ, or the annihilation rate of the vortex line, is

$$\gamma \propto \text{const}. \tag{41}$$

From equation (39), one can obtain

$$l = \text{const}, \qquad \gamma = 0. \tag{42}$$

It is obvious that vortex lines are relatively at rest when $l = \text{const}$.

5. Conclusion

In the present study, a class of Euler flows of an ideal incompressible liquid is considered. The kinetic helicity of vortex lines is classified by the Hopf index, Brouwer degree and linking number in geometry. A mechanism of generation and annihilation of vortex lines is given in section 3. The evolution equation of a vortex line has been given and its bifurcation behavior at the bifurcation points is also discussed in detail in section 4. We give three kinds of length scales in the neighborhood of the singularity point, i.e., $l \propto (t - t^*)^{1/2}, l \propto t - t^*$, $l = \text{const}$. It is obvious that length scales in the branching case are different from those in the the bifurcation case. Because length scales are obtained in the neighborhood of the singularity point, the relations are correct locally. These are different from the length scales in statistical measurement.

Since the kinetic helicity Γ is invariant for the barotropic inviscid flow under conservative body forces, the sum of the final vortex topological number must be equal to that of the original vortex lines at the bifurcation points. This relation and the critical condition determine the bifurcation situation of the vortex lines. The bifurcation behavior becomes complicated for the entangledness of the vortex lines.

Finally, it should be pointed out that in this paper we discussed the bifurcation of vortex lines in Euler flows of an ideal incompressible liquid. In many other cases fluid has viscosity, and are governed by the Navier–Stokes equation. The basic energy estimate shows that for fixed initial data, smooth Navier–Stokes equations converge to a solution of the Euler equation as the viscosity tends to zero. In our method, the bifurcation of vortex in Navier–Stokes flow will also appear. But the A is no longer constant in equation (7), i.e., A is varied because of dissipation. I hope that this case may be discussed in further papers.

Acknowledgments

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Appendix A

Under the regular condition

$$\mathrm{D}\left(\frac{\psi}{x}\right) \neq 0, \tag{A.1}$$

the general solution of

$$\psi^1(t, x^1, x^2, x^3) = 0, \qquad \psi^2(t, x^1, x^2, x^3) = 0 \tag{A.2}$$

is just the vortex line. The kth vortex line L_k can be expressed by the line parameter s:

$$x_k^1 = x_k^1(t, s), \qquad x_k^2 = x_k^2(t, s), \qquad x^3 = x_k^3(t, s). \tag{A.3}$$

The δ-function theory [28] tells us that

$$\delta^2(\boldsymbol{\psi}) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\mathbf{x}(s))}{\left\|D\left(\frac{\psi}{u}\right)\right\|_{M_k}} \,ds, \tag{A.4}$$

where

$$D\left(\frac{\psi}{u}\right) = \frac{1}{2}\epsilon^{ij}\epsilon^{ab}\frac{\partial\psi^a}{\partial u^i}\frac{\partial\psi^b}{\partial u^j}, \qquad i, j = 1, 2, \quad a, b = 1, 2, \tag{A.5}$$

and M_k is the kth planar element transverse to the vortex line L_k with local coordinates (u^1, u^2). The positive integer number β_k is the Hopf index, which means that when $\mathbf{x}$ covers the zero point once, the vector parameter field $\boldsymbol{\psi}$ covers the corresponding region in ψ space β_k times. In Moffatt's paper [7], β_k is also called winding number traced from Gauss. The direction of vector vortex line is given by

$$\frac{dx^i}{ds} = \frac{D^i\left(\frac{\psi}{x}\right)}{D\left(\frac{\psi}{u}\right)}. \tag{A.6}$$

Then from equations (A.4) and (A.6), the transverse field $\boldsymbol{\Omega}$ can be written as

$$\Omega^i = 8\pi A \sum_{k=1}^{N} \beta_k \eta_k \int_{L_k} \frac{dx^i}{ds}\delta^3(\mathbf{x} - \mathbf{x}_k(s))\,ds, \tag{A.7}$$

where $\eta_k = \operatorname{sgn} D\left(\frac{\psi}{u}\right) = \pm 1$. It is the Brouwer degree of the ψ mapping, which characterizes the direction of the vortex line.

Hence,

$$\int_{M_k} \Omega^i \,d\sigma_i = 8\pi A \beta_k \eta_k. \tag{A.8}$$

It is just equation (17).

Appendix B

Linking numbers are the simplest topological relation between two closed curves; this number is zero for two un-linked curves. In order to discuss the linking numbers of the knotted vortex lines, we define the Gauss mapping:

$$\tilde{\mathbf{n}} : S^1 \times S^1 \to S^2, \tag{B.1}$$

where $\tilde{\mathbf{n}}$ is a unit vector

$$\tilde{\mathbf{n}}(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}_k - \mathbf{x}_l}{\|\mathbf{x}_k - \mathbf{x}_l\|}, \tag{B.2}$$

where $\mathbf{x}_l$ and $\mathbf{x}_k$ are the two points, respectively, on the knotted vortex lines ξ_l and ξ_k. When $\mathbf{x}_l$ and $\mathbf{x}_k$ are the same point on the same vortex line ζ, $\tilde{\mathbf{n}}$ is just the unit tangent vector. When $\mathbf{x}_l$ and $\mathbf{x}_k$ cover the corresponding vortex lines ξ_j and ξ_k, $\tilde{\mathbf{n}}$ becomes the section of the sphere bundle S^2. As in the above section, we can define two two-dimensional unit vectors $\tilde{\mathbf{e}} = \tilde{e}(\mathbf{x}_l, \mathbf{x}_k)$. $\tilde{\mathbf{e}}$, $\tilde{\mathbf{n}}$ are normal to each other, i.e.,

$$\begin{aligned} \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2 &= \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{n}} = \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{n}} = 0, \\ \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_1 &= \tilde{\mathbf{e}}_2 \cdot \tilde{\mathbf{e}}_2 = \tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1. \end{aligned} \tag{B.3}$$

In fact, the velocity $\mathbf{V}$ can be expressed as

$$V_i = \mathbf{2}A\epsilon^{ab}e^a\partial_i e^b, \qquad a, b = 1, 2. \tag{B.4}$$

Substituting it into equation (19), one can obtain a new expression of the kinetic helicity

$$\Gamma = 16\pi A^2 \sum_{k=1}^{N} \beta_k \eta_k \oint_{\xi_k} \epsilon^{ab}\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_i\, \mathrm{e}^b(\mathbf{x}, \mathbf{y})\, \mathrm{d}x^i. \tag{B.5}$$

It can also be written as

$$\Gamma = 16\pi A^2 \sum_{k,l=1}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_j\, \mathrm{e}^b(\mathbf{x}, \mathbf{y})\, \mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{B.6}$$

There are three cases: (1) ξ_k, ξ_l are different vortex lines, and $\mathbf{x}_l, \mathbf{x}_k$ are different points; (2) ξ_k, ξ_l are the same vortex line, and $\mathbf{x}, \mathbf{y}$ are different points; (3) ξ_k, ξ_l are the same vortex line, and $\mathbf{x}_l, \mathbf{x}_k$ are same point. Thus, equation (B.6) can be written as

$$\begin{aligned}\Gamma = 64\pi^2 A^2 \Bigg\{ &\frac{1}{4\pi} \sum_{k=1(\mathbf{x}\neq\mathbf{y})}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_k} \epsilon^{ab} \partial_i\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_j\, \mathrm{e}^b(\mathbf{x}, \mathbf{y})\, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &+ \frac{1}{4\pi} \sum_{k=1(\mathbf{x}=\mathbf{y})}^{N} \beta_k \eta_k \oint_{\xi_k} \epsilon^{ab} \partial_i\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_j\, \mathrm{e}^b(\mathbf{x}_l, \mathbf{x}_k)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \\ &+ \frac{1}{4\pi} \sum_{k,l=1}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_j\, \mathrm{e}^b(\mathbf{x}_l, \mathbf{x}_k)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \Bigg\}.\end{aligned} \tag{B.7}$$

The first term is just the writhing number [29] $w_r(\xi_k)$ of vortex line ξ_k. The second term is the twisting number $T_w(\xi_k)$ of the vortex line ξ_k. From White's formula [30], the self-linking number $S(\xi_k)$ of the vortex line ξ_k is given as follows:

$$S(\xi_k) = w_r(\xi_k) + T_w(\xi_k). \tag{B.8}$$

The third term is the Gauss linking number L of the vortex lines ξ_k and ξ_l, i.e.,

$$L(\xi_k, \xi_l) = \frac{1}{4\pi} \sum_{l=1}^{N} \beta_k \eta_k \oint_{\xi_k} \oint_{\xi_l} \epsilon^{ab} \partial_i\, \mathrm{e}^a(\mathbf{x}_l, \mathbf{x}_k)\partial_j\, \mathrm{e}^b(\mathbf{x}_l, \mathbf{x}_k)\, \mathrm{d}x^i \wedge \mathrm{d}x^j, \qquad k \neq l. \tag{B.9}$$

We then obtain the important result

$$\Gamma = 64\pi^2 A^2 \left[\sum_{k=1}^{N} \beta_k \eta_k S(\xi_k) + \sum_{k,l=1}^{N} \beta_k \eta_k L(\xi_k, \xi_l) \right], \tag{B.10}$$

which is equation (20).

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